3601. The shortest path between a point and a curve lies along a normal to the curve. At $(p, 1/p)$, the equation of the normal is

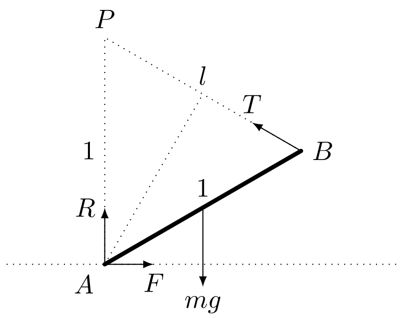
$$y = p^2x - p^3 + \tfrac{1}{p}.$$

Substituting $(0,0)$ gives $p = \pm 1$. The other point $(-0.8, 1.675)$ produces the quartic equation

$$1.675 = -0.8p^2 - p^3 + \tfrac{1}{p}.$$

The relevant root is $p = 0.5$. So, the distances to the hyperbola are ≈ 1.41 from $(0,0)$ and ≈ 1.34 from $(-0.8, 1.675)$. Hence, the latter is closer.

3602. The force diagram is



$\triangle BAP$ is isosceles. So, let $\angle BAP = 2\theta$. Then $\sin\theta = \frac{1}{2}l$. The angle of inclination of the tension is also θ. Taking moments around A,

$$\begin{aligned} & mg \cdot \tfrac{1}{2}\cos(90° - 2\theta) = T\cos\theta \\ \Longrightarrow\ & \tfrac{1}{2}mg\sin 2\theta = T\cos\theta \\ \Longrightarrow\ & mg\sin\theta\cos\theta = T\cos\theta. \end{aligned}$$

Since $\cos\theta \neq 0$, we can divide through by it, which leaves $T = mg\sin\theta = \frac{1}{2}mgl$, as required.

3603. (a) The highest terms are ..., 512, 1024. Hence, two different terms can only sum to greater than 1000 if one of them is $2^{10} = 1024$. There are 9 successful pairs, giving a probability $\frac{9}{45} = \frac{1}{5}$.

(b) The lowest terms are $1, 2, 4, 8, \ldots$. Anything above these, and the sum is greater than 15. Any combination of these four yields a sum less than 15. So, there are ${}^4\mathrm{C}_2 = 6$ successful outcomes, giving probability $\frac{6}{45} = \frac{2}{15}$.

3604. (a) We want to express the numerator as

$$9x^2 \equiv A(3x+1)^2 + B(3x+1) + C.$$

Equating coefficients, we need $A = 1$, $B = -2$ and $C = 1$. This gives

$$\frac{9x^2}{3x+1} \equiv 3x - 1 + \frac{1}{3x+1}.$$

— Alternative Method —

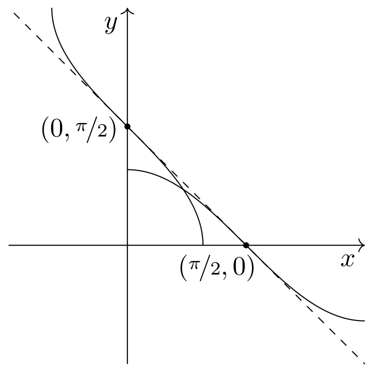
The numerator can be written as $9x^2 - 1 + 1$, which is $(3x+1)(3x-1) + 1$. This gives

$$\frac{9x^2}{3x+1} \equiv 3x - 1 + \frac{1}{3x+1}.$$

(b) Integrating term by term,

$$\begin{aligned} & \int 3x - 1 + \frac{1}{3x+1}\,dx \\ &= \tfrac{3}{2}x^2 - x + \tfrac{1}{3}\ln|3x+1| + c. \end{aligned}$$

3605. The graph $y = \arccos x$ is the reflection of (a part of) the graph $y = \cos x$ in $y = x$. So, to find the linear approximation to $y = \arccos x$ at $(0, \pi/2)$, we find the linear approximation to $y = \cos x$ at $(\pi/2, 0)$.



The tangent line at this point is $y = \frac{\pi}{2} - x$. Since this has gradient -1, reflecting it in $y = x$ leaves it unchanged. Both are the dashed line shown above. Hence, for small x in radians, $\arccos x \approx \frac{\pi}{2} - x$.

3606. We calculate the probabilities of the values of n.

- For $n = 1$, the outcomes alternate as HTHT or THTH, giving $p_1 = \frac{2}{16} = \frac{1}{8}$.
- For $n = 4$, the outcomes are HHHH and TTTT, so p_4 is also $\frac{1}{8}$.
- For $n = 3$, the possibilities are HHHT, TTTH, THHH and HTTT, giving $p_3 = \frac{1}{4}$.
- p_2 is then $1 - p_1 - p_3 - p_4 = \frac{1}{2}$.

So, the probability distribution is

n	1	2	3	4
p_n	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

The average value is the expectation, which is

$$\begin{aligned} \sum np_n &= 1 \cdot \tfrac{1}{8} + 2 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{4} + 4 \cdot \tfrac{1}{8} \\ &= 2.375. \end{aligned}$$

3607. Rotation clockwise around the origin by 90° is equivalent to

① reflection in $y = x$, followed by
② reflection in the x axis.

Performing these algebraically,

① $y = x^3 - x^2$ becomes $x = y^3 - y^2$,

② we replace y by $-y$, giving $x = -y^3 - y^2$.

3608. (a) This doesn't hold: a counterexample is $a = 2$ and $b = -1$.

(b) This doesn't hold: same counterexample.

(c) This holds: for positive numbers $|a|$ and $|b|$, reciprocating reverses the order of magnitude.

3609. The area is given by four times that beneath a quarter circle:

$$A = 4\int_0^r \sqrt{r^2 - x^2}\,dx.$$

Let $x = r\sin\theta$. This gives $dx = r\cos\theta\,d\theta$. The limits are $\theta = 0$ to $\theta = \frac{\pi}{2}$. Enacting the substitution,

$$\begin{aligned} A &= 4\int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta} r\cos\theta\,d\theta \\ &= 4r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta. \end{aligned}$$

Using a double-angle formula,

$$\begin{aligned} A &= 2r^2\int_0^{\frac{\pi}{2}} 1 + \cos 2\theta\,d\theta \\ &= 2r^2\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\frac{\pi}{2}} \\ &= 2r^2 \cdot \tfrac{\pi}{2} \\ &= \pi r^2, \text{ as required.} \end{aligned}$$

Nota Bene

This proof is (pardon the pun) somewhat circular, as the values of the trig functions are originally defined by the circle itself, i.e. by the quantity π. Nevertheless, it is reassuring to know that such a proof works!

3610. This is a cubic in $\sqrt{x}$. Let $z = \sqrt{x}$. Then we have

$$z^3 - 6z^2 + 12z - 8 = 0.$$

Using a polynomial solver, $z = 2$. So, $\sqrt{x} = 2$, which gives $x = 4$.

3611. Let $z = x^2 - y^2$. Substituting for x,

$$z = 4y - y^2.$$

Setting $\frac{dz}{dy}$ to zero for SPs, $4 - 2y = 0$, which gives $y = 2$. Substituting back in, the maximum value of $x^2 - y^2$ is 4.

3612. (a) The integral of the first term is $\big[x\big]_a^{2a}$, which is a. The integral of the second term is $-b$, because $[a, 2a]$ is the rotated (thus negative-valued) image of $[0, a]$. Hence, the value of the integral is $a - b$.

(b) The integral of the first term is $\big[x\big]_0^{2a}$, which is $2a$. The integral of the second term is zero, as $[0, a]$ produces b, while $[a, 2a]$ produces $-b$. Hence, the value of the integral is $2a$.

3613. (a) The rate peaks when the derivative of the rate is zero. Using the product rule,

$$\begin{aligned} \frac{d^2m}{dt^2} &= e^{p-\frac{1}{q}t} - \tfrac{1}{q}te^{p-\frac{1}{q}t} \\ &= e^{p-\frac{1}{q}t}(1 - \tfrac{1}{q}t). \end{aligned}$$

The exponential term cannot be zero, so the rate peaks at $(1 - \frac{1}{q}t) = 0$, which is $t = q$. The value of the rate at this point is

$$\frac{dm}{dt} = qe^{p-1}.$$

(b) The rate is $\frac{dm}{dt}$. The integral is the continuous sum of this rate, which gives the total quantity. Taking limits $t = 0$ and $t = T$, the maximum m_{max} is given by the limit at $T \to \infty$, which gives the proposed definite integral.

(c) We integrate by parts. The ingredients are

$$u = t, \quad \dot{v} = e^{p-\frac{1}{q}t}, \quad \dot{u} = 1, \quad v = -qe^{p-\frac{1}{q}t}.$$

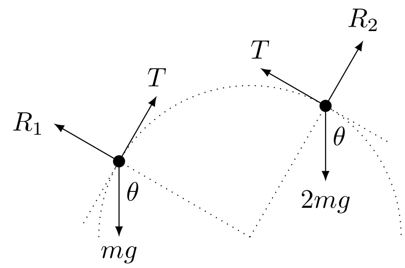
The parts formula gives

$$\begin{aligned} m_{\text{total}} &= \Big[-qte^{p-\frac{1}{q}t}\Big]_0^\infty + \int_0^\infty qe^{p-\frac{1}{q}t}\,dt \\ &= \Big[-qte^{p-\frac{1}{q}t} - q^2e^{p-\frac{1}{q}t}\Big]_0^\infty. \end{aligned}$$

The upper bound produces a value of zero, because the exponentials, which tend to zero, dominate. So,

$$\begin{aligned} m_{\text{total}} &= (0) - \big(-q^2e^p\big) \\ &\equiv q^2e^p. \end{aligned}$$

3614. (a) The force diagrams are



(b) Resolving along the circumference,

$$\begin{aligned} &2mg\cos\theta - mg\sin\theta = 0 \\ \Longrightarrow\ &\tan\theta = 2. \end{aligned}$$

3615. Differentiating by the chain rule,

$$\frac{d(\operatorname{cosec}\theta)}{d\theta} = -(\sin\theta)^{-2}\cos\theta.$$

Using a double-angle formula, the RHS is

$$\begin{aligned}&\frac{2\cos\theta}{\cos 2\theta - 1}\\ \equiv\ &\frac{2\cos\theta}{(1-2\sin^2\theta - 1}\\ \equiv\ &\frac{\cos\theta}{-\sin^2\theta}\\ \equiv &-(\sin\theta)^{-2}\cos\theta.\end{aligned}$$

This proves the identity.

3616. (a) For $X \geq 9$, we require $N \geq 9$. The successful outcomes (N, X) are

$$\begin{aligned}&(9,9), \text{ with probability } \tfrac{1}{10}\times\tfrac{1}{9},\\ &(10,9), \text{ with probability } \tfrac{1}{10}\times\tfrac{1}{10},\\ &(10,10), \text{ with probability } \tfrac{1}{10}\times\tfrac{1}{10}.\end{aligned}$$

So, $\mathbb{P}(X \geq 9) = \frac{1}{90} + \frac{1}{100} + \frac{1}{100} = \frac{7}{225}$.

(b) Considering the above as the possibility space,

$$\mathbb{P}(N = 10 \mid X \geq 9) = \frac{\frac{1}{100}+\frac{1}{100}}{\frac{1}{90}+\frac{1}{100}+\frac{1}{100}} = \frac{9}{14}.$$

3617. Factorising, we have $y = (1-x)^3(1+x)^3$. This is a negative sextic, with triple roots at $x = \pm 1$. The proposed curve is also symmetrical in the y axis, which it crosses at $(0, 1)$. All of these facts agree with the graph shown.

3618. Consider first an equilateral triangle of side length 33 cm. This has perpendicular height $33\sqrt{3}/2$. Its centre divides this height in the ratio $1 : 2$, so the distance from the centre to a vertex is

$$33\tfrac{\sqrt{3}}{2} \times \tfrac{2}{3} \approx 19.05.$$

Since this is greater than 19, the $(33, 33, 33)$ cm equilateral triangle won't fit. Hence, the larger $(33, 34, 35)$ cm triangle won't fit either.

3619. (a) Direct integration gives

$$\int_{-1}^{1} x^3 - x\,dx = \Big[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\Big]_{-1}^{1} = 0.$$

For the trapezium rule, with four strips each of width $1/2$, we calculate the value of $x^3 - x$ at $x = -1, -1/2, 0, 1/2, 1$, giving $0, 3/8, 0, -3/8, 0$. The full calculation is

$$\tfrac{1}{2}\cdot\tfrac{1}{2}\Big(0 + 2\big(\tfrac{3}{8} + 0 - \tfrac{3}{8}\big) + 0\Big) = 0.$$

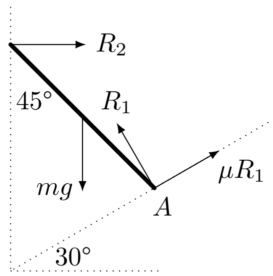
(b) The curve $y = x^3 - x$ has rotational (odd) symmetry around O. So, the signed areas over $[-1, 0]$ and $[0, 1]$ cancel in the integration, which gives a total signed area of zero.

The same cancellation happens when using the trapezium rule: it too calculates signed areas.

3620. Let $z = e^x$ and $y = \ln x$, so $\frac{dz}{dx} = e^x$ and $\frac{dy}{dx} = \frac{1}{x}$. Then the chain rule gives

$$\frac{d(e^x)}{d(\ln x)} = \frac{dz}{dy} = \frac{dz}{dx} \div \frac{dy}{dx} = xe^x, \text{ as required.}$$

3621. (a) The force diagram, for limiting friction, is



(b) i. $R_1\cos 30° + \mu R_1 \sin 30° - mg = 0$,

ii. $R_2 + \mu R_1 \cos 30° - R_1 \sin 30° = 0$,

iii. $mg\cos 45° - R_2 \cdot 2\sin 45° = 0$.

(c) The moments give $R_2 = \frac{1}{2}mg$. Substituting this in, multiplying each equilibrium equation by 2 and simplifying,

$$\begin{aligned}R_1(\sqrt{3}+\mu) &= 2mg,\\ R_1(1-\sqrt{3}\mu) &= mg.\end{aligned}$$

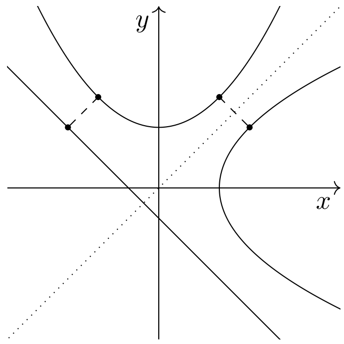
Dividing these equations,

$$\begin{aligned}&\frac{\sqrt{3}+\mu}{1-\sqrt{3}\mu} = 2\\ \Longrightarrow\ &\sqrt{3}+\mu = 2 - 2\sqrt{3}\mu\\ \Longrightarrow\ &\mu(2\sqrt{3}+1) = 2-\sqrt{3}\\ \Longrightarrow\ &\mu = \frac{2-\sqrt{3}}{2\sqrt{3}+1}\\ &\quad = \frac{5\sqrt{3}-8}{11}, \text{ as required.}\end{aligned}$$

— Nota Bene —

This coefficient of friction is small: $\mu \approx 0.06$. This means that the ladder is almost in its most stable position on such a slope.

3622. Together, the three loci are symmetrical in the line $y = x$, dotted below:



The shortest distances lie along the dashed lines. These have gradients ± 1. So, we set the derivative of $y = \frac{1}{2}x^2 + 1$ to ± 1. This gives $x = \pm 1$. Hence, the coordinates of the marked points are $(\pm 3/2, 1)$ and $(\pm 1, 3/2)$. Symmetry dictates that the loci are equidistant.

3623. (a) In partial fractions, the integrand is

$$\frac{2}{x^2-1} \equiv \frac{1}{x-1} - \frac{1}{x+1}.$$

We can now integrate:

$$\begin{aligned} I &= \int \frac{1}{x-1} - \frac{1}{x+1}\, dx \\ &= \ln|x-1| - \ln|x+1| + c \\ &\equiv \ln\left|\frac{x-1}{x+1}\right| + c. \end{aligned}$$

Exponentiating this,

$$\begin{aligned} e^I &= e^{\ln\left|\frac{x-1}{x+1}\right| + c} \\ &\equiv e^{\ln\left|\frac{x-1}{x+1}\right|} e^c \\ &\equiv \left|\frac{x-1}{x+1}\right| e^c \\ &= A\left|\frac{x-1}{x+1}\right|, \text{ as required.} \end{aligned}$$

(b) The constant of integration c can take any value. Since $A = e^c$, the constant A must be positive: $A > 0$.

3624. It is possible for the tests to yield different results. In the two-tailed test, the two critical regions are each smaller than the equivalent one-tailed region, corresponding to e.g. 2.5% probability rather than 5% probability. So, it is possible that the larger one-tailed critical region contains the test statistic, thereby providing sufficient evidence for (a specific direction of) correlation.

3625. The area of the segment is given by

$$\begin{aligned} A &= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2 \sin\theta \\ &\equiv \tfrac{1}{2}r^2(\theta - \sin\theta). \end{aligned}$$

Differentiating implicitly with respect to time,

$$\frac{dA}{dt} = \tfrac{1}{2}r^2\left(1 - \cos\theta \tfrac{d\theta}{dt}\right).$$

We are told that the angle is increasing at 1 radian per second, so $\frac{d\theta}{dt} = 1$. This gives

$$\frac{dA}{dt} = \tfrac{1}{2}r^2(1 - \cos\theta).$$

With θ small (and defined in radians), we can use the approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Hence,

$$\begin{aligned} \frac{dA}{dt} &\approx \tfrac{1}{2}r^2\Big(1 - \big(1 - \tfrac{1}{2}\theta^2\big)\Big) \\ &\equiv \tfrac{1}{4}r^2\theta^2, \text{ as required.} \end{aligned}$$

3626. Differentiating with respect to y,

$$\begin{aligned} x &= \frac{25}{16+y^2} \\ \Longrightarrow \frac{dy}{dx} &= \frac{-50y}{(16+y^2)^2}. \end{aligned}$$

Setting $y = 4$, this gives

$$\frac{dx}{dy} = -\frac{25}{128}.$$

Also $x = \frac{25}{32}$. So, the equation of the tangent is

$$x = \frac{25}{16} - \frac{25}{128}y.$$

Solving for re-intersections,

$$\begin{aligned} \frac{25}{16} - \frac{25}{128}y &= \frac{25}{16+y^2} \\ \Longrightarrow (y-4)^2 y &= 0. \end{aligned}$$

This shows the point of tangency at $y = 4$, and a point of intersection on the x axis, as required.

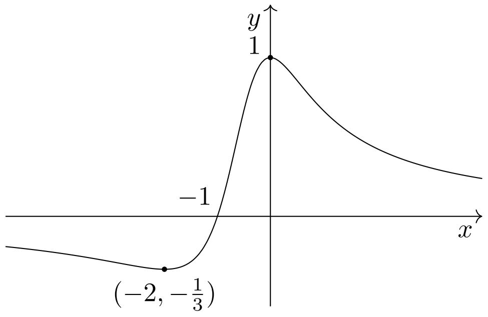
3627. The generation of roots beyond the first has been left too late. They should appear in the second line, as the tan function is undone. Because of subsequent division by 2, the student should have found four roots:

$$\begin{aligned} &\tan(2\theta + 15^\circ) = 1 \\ \therefore\ &2\theta + 15^\circ = 45^\circ, 225^\circ, 405^\circ, 585^\circ \\ \Longrightarrow\ &2\theta = 30^\circ, 210^\circ, 390^\circ, 570^\circ \\ \Longrightarrow\ &\theta \in \{15^\circ, 105^\circ, 195^\circ, 285^\circ\}. \end{aligned}$$

3628. The denominator has $\Delta = -3$, so the curve has no vertical asymptotes. It crosses the x axis at $(-1, 0)$ and the y axis at $(0, 1)$. Looking for SPs, we use the quotient rule:

$$\begin{aligned} &\frac{(x^2+x+1) - (x+1)(2x+1)}{(x^2+x+1)^2} = 0 \\ \Longrightarrow\ &(x^2+x+1) - (x+1)(2x+1) = 0 \\ \Longrightarrow\ &x = 0, -2. \end{aligned}$$

So, there are SPs at $(0, 1)$ and $(-2, -1/3)$. Also, since the degree of the denominator is larger than the degree of the numerator, $y \to 0$ as $x \to \pm\infty$. Putting all of the above together, the curve is



3629. Consider the trilogy as a single object. Then there are eight objects to arrange in 8! ways. We may then rearrange the trilogy amongst itself, in any of 3! ways. This gives $3! \times 8! = 241920$ ways.

3630. Multiplying up by the denominators, we require

$$\begin{aligned} \frac{164}{6283} &= \frac{1}{p} + \frac{1}{q} \\ \Longrightarrow 2^2 \cdot 41pq &= 61 \cdot 103(p+q). \end{aligned}$$

The RHS has (single) factors of 61 and 103. Hence, so does the LHS. These cannot be in 2^2 or 41, so must be in pq. And, since 61 and 103 are prime, each must appear in only one of p and q. Hence, the only possibility, which we can verify, is

$$\frac{164}{6283} = \frac{1}{61} + \frac{1}{103}.$$

3631. (a) $\dfrac{1}{r(r+1)} \equiv \dfrac{1}{r} - \dfrac{1}{r+1}$.

(b) Writing the sum longhand,

$$\begin{aligned} &\sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+1} \\ &\equiv \left(\frac{1}{1} - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{3}\right) - ... - \left(\frac{1}{n} - \frac{1}{n+1}\right). \end{aligned}$$

All but the two outermost terms cancel, which allows for calculation of the sum.

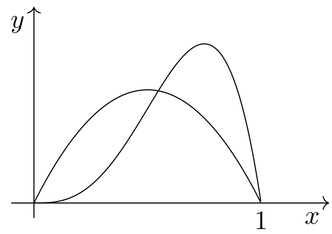
(c) The above shows that

$$\begin{aligned} \sum_{r=1}^{n} \frac{1}{r(r+1)} &\equiv 1 - \frac{1}{n+1} \\ &\equiv \frac{n}{n+1}, \text{ as required.} \end{aligned}$$

3632. (a) Factorising, the curves are

$$\begin{aligned} y &= x(1-x), \\ y &= x^3(1-x). \end{aligned}$$

These both have roots at $x = 0$ and $x = 1$. In $y = \mathrm{f}_2(x)$, these are both single roots: the curve is a parabola. In $y = \mathrm{f}_4(x)$, the root at $x = 0$ is a triple root, which is a (cubic) point of tangency. Since the areas below the curve must be equal, the turning point of the quartic must be higher than the turning point of the quadratic:



(b) For area 1, we require

$$\begin{aligned} \int_0^1 k_n\left(x^{n-1} - x^n\right) &= 1 \\ \Longrightarrow k_n\left[\tfrac{1}{n}x^n - \tfrac{1}{n+1}x^{n+1}\right]_0^1 &= 1 \\ \Longrightarrow k_n\left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) &= 1 \\ \Longrightarrow k_n\left(\tfrac{1}{n(n+1)}\right) &= 1 \\ \Longrightarrow k_n &= n(n+1), \text{ as required.} \end{aligned}$$

3633. Rearranging to make y the subject,

$$\begin{aligned} (x+y)^3 - x &= 0 \\ \Longrightarrow x + y &= x^{\frac{1}{3}} \\ \Longrightarrow y &= x^{\frac{1}{3}} - x. \end{aligned}$$

Setting $y = 0$, the intercepts are $x = 0, \pm 1$. So, we calculate

$$\int_0^1 x^{\frac{1}{3}} - x \, dx = \left[\tfrac{3}{4}x^{\frac{4}{3}} - \tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{4}.$$

Since $y = x^{\frac{1}{3}} - x$ contains only odd powers of x, the curve has rotational symmetry around $(0, 0)$. Hence, the area enclosed on $[-1, 0]$ is also $\frac{1}{4}$.

3634. Using a polynomial solver,

$$\begin{aligned} 48x^3 - 340x^2 + 802x - 630 &= 0 \\ \Longrightarrow x &= \tfrac{9}{4}, \tfrac{7}{3}, \tfrac{5}{2}. \end{aligned}$$

So, by the factor theorem, the polynomial must have factors $(4x - 9)$, $(3x - 7)$ and $(2x - 5)$. Since the leading coefficients of these multiply to 24, we also need a factor of 2. So, the factorisation is

$$\begin{aligned} &48x^3 - 340x^2 + 802x - 630 \\ &\equiv 2(4x-9)(3x-7)(2x-5). \end{aligned}$$

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3635. Multiplying out, we get $x^2y + xy^2 = 2$. This may be seen as a quadratic in y. Using the formula,

$$\begin{aligned} xy^2 + x^2y - 2 &= 0 \\ \implies y &= \frac{-x^2 \pm \sqrt{x^4+8x}}{2x} \\ &= -\frac{x}{2} \pm \sqrt{\frac{x^2}{4} + \frac{2}{x}}, \text{ as required.} \end{aligned}$$

3636. We calculate various sums:

- of the first 100 integers: 5050,
- of the first 25 multiples of 4: 1300,
- of the first 20 multiples of 5: 1050,
- of the first 5 multiples of 20: 300.

Subtracting 1300 and 1050 from 5050, we subtract the multiples of 20 twice, so we add them back on. This gives $5050 - 1300 - 1050 + 300 = 3000$.

3637. (a) The large block will not move.

The pulleys are smooth, so the forces exerted on each pulley by the string are symmetrical: the string applies $\sqrt{2}T$ to each pulley, at 45° below the horizontal. When the large block and pulleys are considered as one system, the horizontal components of these forces cancel, so the resultant horizontal force on the larger block is zero.

(b) The string will accelerate towards the block of mass m_2. In this direction, the equation of motion along the string is

$$\begin{aligned} m_2g - m_1g &= (m_1+m_2)a \\ \implies a &= \frac{(m_2-m_1)g}{m_1+m_2}, \text{ as required.} \end{aligned}$$

3638. We can rewrite as

$$\begin{aligned} y &= \frac{1-x}{1+x} \\ &\equiv \frac{2-(1+x)}{1+x} \\ &\equiv 2 - \frac{1}{1+x}. \end{aligned}$$

This is a transformation of $y = 1/x$ by reflection and translation. Neither affects the existence of points of inflection. Hence, since $y = 1/x$, which is the standard reciprocal graph, has no points of inflection, neither does this graph.

3639. (a) The successful outcomes are $(0,3)$, $(1,2)$ and vice versa. So,

$$\begin{aligned} &\mathbb{P}(X_1 + X_2 = 3) \\ &= 2\left(\tfrac{1}{8}\cdot\tfrac{1}{8} + \tfrac{3}{8}\cdot\tfrac{3}{8}\right) \\ &= \tfrac{5}{16}. \end{aligned}$$

(b) Since X_1 and X_2 are independent, their sum can be considered as a single binomial variable $Y = X_1 + X_2$, for which $n = 6$.

3640. Let $u = x^3$ and $v' = x^2\sqrt{x^3+1}$. Then $u' = 3x^2$. To find v, we integrate by inspection:

$$\begin{aligned} v &= \int x^2\sqrt{x^3+1}\,dx \\ &\equiv \tfrac{1}{3}\int 3x^2\sqrt{x^3+1}\,dx \\ &= \tfrac{1}{3}\cdot\tfrac{2}{3}\left(x^3+1\right)^{\frac{3}{2}} \\ &\equiv \tfrac{2}{9}\left(x^3+1\right)^{\frac{3}{2}}. \end{aligned}$$

We need no $+c$ in the above: parts only requires a $+c$ at the end. Substituting into the parts formula,

$$\begin{aligned} I &= \int x^5\sqrt{x^3+1}\,dx \\ &= x^3\cdot\tfrac{2}{9}\left(x^3+1\right)^{\frac{3}{2}} - \int 3x^2\cdot\tfrac{2}{9}\left(x^3+1\right)^{\frac{3}{2}}\,dx. \end{aligned}$$

We integrate by inspection again, giving

$$\begin{aligned} I &= \tfrac{2}{9}x^3\left(x^3+1\right)^{\frac{3}{2}} - \tfrac{2}{9}\cdot\tfrac{2}{5}\left(x^3+1\right)^{\frac{5}{2}} + c \\ &\equiv \tfrac{2}{9}x^3\left(x^3+1\right)^{\frac{3}{2}} - \tfrac{4}{45}\left(x^3+1\right)^{\frac{5}{2}} + c. \end{aligned}$$

This may also be expressed as

$$I = \tfrac{2}{45}\left(1+x^3\right)^{\frac{3}{2}}\left(3x^3-2\right) + c.$$

3641. We write the LHS in harmonic form: $R\sin(t-\alpha)$. So, we require

$$R\sin t\cos\alpha - R\cos t\sin\alpha \equiv 2\sin t - 2\sqrt{3}\cos t.$$

Equating coefficients, $R\cos\alpha = 2$, $R\sin\alpha = 2\sqrt{3}$. Taking the Pythagorean sum, $R = 4$. Dividing the equations gives $\tan\theta = \sqrt{3}$, so $\theta = \frac{\pi}{3}$. Solving,

$$\begin{aligned} 4\sin\left(t - \tfrac{\pi}{3}\right) &= \sqrt{6} + \sqrt{2} \\ \therefore\quad t - \tfrac{\pi}{3} &= \tfrac{5\pi}{12}, \tfrac{7\pi}{12} \\ \implies t &= \tfrac{3\pi}{4}, \tfrac{11\pi}{12}. \end{aligned}$$

3642. The differences between the times are $36 - 12 = 24$ and $60 - 36 = 24$. Hence, since the population is growing exponentially, the population scale factor is the same over $[12, 36]$ as it is over $[36, 60]$. This gives

$$a = 2304 \times \frac{2304}{576} = 9216.$$

3643. Using the sum-product identities, the RHS is

$$\begin{aligned} &\frac{\sin 2\theta + \sin 2\phi}{\cos 2\theta + \cos 2\phi} \\ \equiv\ &\frac{2\sin\left(\frac{2\theta+2\phi}{2}\right)\cos\left(\frac{2\theta-2\phi}{2}\right)}{2\cos\left(\frac{2\theta+2\phi}{2}\right)\cos\left(\frac{2\theta-2\phi}{2}\right)} \\ \equiv\ &\frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} \\ \equiv\ &\tan(\theta+\phi), \text{ as required.} \end{aligned}$$

3644. (a) The velocity increases linearly for $t \in [0, 4]$, so it is maximised in that domain at $t = 4$, $v = 20$. For $t \geq 4$, the maximum velocity is at $-4t+16 = 0$, which is $t = 4$. So, the boundary between the two domains is the vertex of the parabola. The global maximum velocity is 20 units per second.

(b) To find the displacement, we integrate velocity piecewise. After 4 seconds, the displacement is

$$\int_0^4 3t + 8\,dt = 56.$$

For $t \in [4, 10]$, the velocity is positive initially, and then becomes negative. So, the greatest negative displacement from position at $t = 4$ occurs at $t = 10$.

$$\int_4^{10} -2t^2 + 16t - 12\,dt = -24.$$

Since $56 - 24 > 0$, the particle does not return to its original position in the first ten seconds.

3645. This is a quadratic in $\ln x$:

$$\begin{aligned} &\frac{(\ln x)^2 + 12(\ln 2)^2}{\ln x} = \ln 128 \\ \Longrightarrow\ &(\ln x)^2 + 12(\ln 2)^2 = \ln 128 \ln x \\ \Longrightarrow\ &(\ln x)^2 - 7\ln 2 \ln x + 12(\ln 2)^2 = 0 \\ \Longrightarrow\ &(\ln x - 3\ln 2)(\ln x - 4\ln 2) = 0 \\ \Longrightarrow\ &\ln x = \ln 8, \ln 16 \\ \Longrightarrow\ &x = 8, 16. \end{aligned}$$

3646. Let the far vertex of the base be the origin, giving A, B, C coordinates $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. The centre of $\triangle ABC$ is then $(1/3, 1/3, 1/3)$. The distance from this point to the origin is $\sqrt{3}/3$. The area of $\triangle ABC$ is $\sqrt{3}/2$. So, the volume of pyramid $OABC$ is $1/6$.

Hence, if two points are chosen at random in the cube, then the number of points lying in this region is $X \sim B(2, 1/6)$. This gives p as

$$\begin{aligned} \mathbb{P}(X = 1) &= 2 \times \tfrac{1}{6} \times \tfrac{5}{6} \\ &= \tfrac{5}{18}, \text{ as required.} \end{aligned}$$

3647. (a) The integrand has denominator $e^x - 1$, which tends to infinity as $x \to \infty$. So, the integrand tends to zero as $x \to \infty$. If such convergence to zero happens *quickly enough*, as it does with exponential decay, then it is possible to get a finite value for

$$\lim_{X\to\infty} \int_0^X \frac{x}{e^x - 1}\,dx.$$

(b) For a four-strip approximation on $[0, 4]$:

x	0	1	2	3	4
$f(x)$	1	0.582	0.313	0.157	0.0746

With strips of width 1, the approximation is

$$\tfrac{1}{2}\big(1 + 2(0.582 + 0.313 + 0.157) + 0.075\big) \approx 1.59.$$

The percentage error is given by

$$\frac{1.59 - \frac{\pi^2}{6}}{\frac{\pi^2}{6}} = -0.033...$$

So, the approximation underestimates the true value of the integral by around 3%.

3648. The endpoints of the chord are $(\pm 2, \mp 10)$. So, the chord is $y = -5x$. Solving simultaneously,

$$\begin{aligned} &x^4 - 4x^2 - 5x = -5x \\ \Longrightarrow\ &x^4 - 4x^2 = 0 \\ \Longrightarrow\ &x^2(x + 2)(x - 2) = 0. \end{aligned}$$

This has single roots at $x = \pm 2$, which are the endpoints of the chord, and a double root between the two endpoints at $x = 0$. This is a point of tangency.

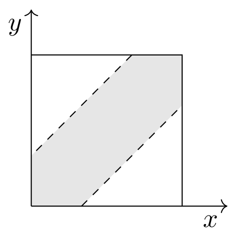
— Alternative Method —

Since the question concerns straight lines (chords and tangents), the $-5x$ term, which is linear, does not affect the result. So, consider

$$\begin{aligned} y &= x^4 - 4x^2 \\ &\equiv x^2(x + 2)(x - 2). \end{aligned}$$

This has roots at $x = \pm 2$, so the chord is the x axis. There is a double root at $x = 0$, so the chord is tangent to the curve, as required.

3649. The possibility space is a unit square. The total area is 1, so area represents probability. The locus of points (x, y) in which $|x - y| < 1/3$ is



Subtracting the unshaded triangles, each of area $2/9$, from the square,

$$\mathbb{P}\left(|x - y|\right) < \tfrac{1}{3} = \tfrac{5}{9}.$$

3650. The line through the origin and the point $\left(a, \frac{1}{2}a^2\right)$ has equation $y = \frac{1}{2}ax$. Solving this simultaneously with the curve,

$$\begin{aligned} \frac{a^2x^2}{a^2+x^2} &= \tfrac{1}{2}ax \\ \implies ax\big(a^2-2ax+x^2\big) &= 0 \\ \implies ax(a-x)^2 &= 0. \end{aligned}$$

Since this has a double root at $x=a$, the tangent at that point passes through the origin.

— Alternative Method —

Differentiating by the quotient rule,

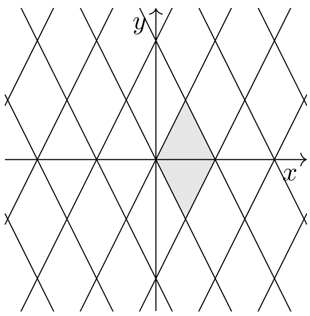
$$\begin{aligned} y &= \frac{a^2x^2}{a^2+x^2} \\ \implies \frac{dy}{dx} &= \frac{2a^2x\big(a^2+x^2\big)-a^2x^2\cdot 2x}{\big(a^2+x^2\big)^2} \\ &\equiv \frac{2a^4x}{\big(a^2+x^2\big)^2}. \end{aligned}$$

So, at $x=a$, the gradient is $a/2$. Also, at $x=a$, the y coordinate is $a^2/2$. Hence, the equation of the tangent at $x=a$ is

$$\begin{aligned} y-\frac{a^2}{2} &= \frac{a}{2}(x-a) \\ \implies y &= \tfrac{1}{2}ax. \end{aligned}$$

This passes through the origin, as required.

3651. The equation $\sin y=\sin 2x$ is satisfied by primary solutions $y=2x$ and $y=\pi-2x$, and any addition of $2n\pi$ to either. Sketching these, the tiling of the plane is



The shaded rhombus has vertices $(0,0)$, $(\pi/2,\pm\pi)$ and $(\pi,0)$. Its area is π^2.

3652. This is false. The equations $y=x$, $y=2x$ and $y=3x$ are three distinct linear equations in two unknowns. Since the three lines are concurrent (at the origin), the solution $(0,0)$ is unique.

3653. (a) Differentiating implicitly by the product rule,

$$\begin{aligned} P &= 4\sqrt{x^2+y^2} \\ \implies \frac{dP}{dt} &= 2\big(x^2+y^2\big)^{-\frac{1}{2}}\big(2x\tfrac{dx}{dt}+2y\tfrac{dy}{dt}\big) \\ &\equiv \frac{4x\frac{dx}{dt}+4y\frac{dy}{dt}}{\sqrt{x^2+y^2}}. \end{aligned}$$

(b) Substituting in the values $x=3$, $y=4$, $\frac{dx}{dt}=2$ and $\frac{dy}{dt}=1$, the rate of change of perimeter is

$$\begin{aligned} \frac{dP}{dt} &= \frac{4\cdot 3\cdot 2+4\cdot 4\cdot 1}{\sqrt{3^2+4^2}} \\ &= 8 \text{ cm per second.} \end{aligned}$$

3654. Using the binomial expansion,

$$\begin{aligned} \sqrt[3]{x^3+h} &\equiv x\left(1+\frac{h}{x^3}\right)^{\frac{1}{3}} \\ &= x\left(1+\frac{h}{3x^3}-\frac{h^2}{9x^6}+...\right) \\ &\equiv x+\frac{h}{3x^2}-\frac{h^2}{9x^5}+... \end{aligned}$$

Substituting this in,

$$\begin{aligned} &\lim_{h\to 0}\frac{\sqrt[3]{x^3+h}-x}{h} \\ \equiv\ &\lim_{h\to 0}\frac{\frac{h}{3x^2}-\frac{h^2}{9x^5}+...}{h} \\ \equiv\ &\lim_{h\to 0}\frac{1}{3x^2}-\frac{h}{9x^5}+... \end{aligned}$$

Every term except the first contains factors of h, so tends to zero. Hence,

$$\lim_{h\to 0}\frac{\sqrt[3]{x^3+h}-x}{h}\equiv\frac{1}{3x^2}.$$

3655. We can place A_1 without loss of generality. Then the probability that A_2 sits next to A_1 is $\frac{2}{5}$. Then, of the ${}^4\mathrm{C}_2=6$ ways the other couples can sit, there are two ways in which each couple sits apart. So, the probability that A_1 and A_2 are the only couple sitting together is $\frac{2}{5}\times\frac{1}{3}$. There are three couples, so we multiply by 3, to give $p=\frac{2}{5}$.

— Alternative Method —

There are 6! ways in which the people can sit. For successful outcomes, there are 3 couples to choose from. There are 6 locations for this couple, and there are 2! orders in which the couple can sit.

For the others, there are 4 people who can sit next to A_1, then 2 people who can sit next to them (must be the other couple), then 1 and 1, giving 8 successful orders of the others.

So, the probability is

$$p=\frac{3\times 6\times 2!\times 8}{6!}=\tfrac{2}{5}.$$

3656. (a) Each pentagon has 5 vertices, and each hexagon 6. This gives $5p+6h$ vertices. But we have counted each vertex three times, as three faces meet at each vertex. So, we divide by the overcounting factor, giving $V = \frac{1}{3}(5p+6h)$.

(b) Using the same logic, since two faces meet at every edge, $E = \frac{1}{2}(5p+6h)$. And $F = p + h$.

(c) We know that $V - E + F = 2$. Subbing in,

$$\begin{aligned} &\tfrac{1}{3}(5p+6h) - \tfrac{1}{2}(5p+6h) + (p+h) = 2 \\ \implies &p = 12. \end{aligned}$$

Hence, there must always be 12 pentagons in such an arrangement. QED.

3657. Setting the output to y, we solve for x:

$$\begin{aligned} y &= \frac{x^2+1}{x^2+4} \\ \implies y(x^2+4) &= x^2+1 \\ \implies (y-1)x^2 + 4y - 1 &= 0 \\ \implies x &= \pm\sqrt{\frac{1-4y}{y-1}}. \end{aligned}$$

Choosing the positive square root, the instruction for the inverse function is

$$\mathrm{g}^{-1} : x \longmapsto \sqrt{\frac{1-4x}{x-1}}.$$

The range of g is $[1/4, 1)$, and g is one-to-one over $[0, \infty)$. Hence, the domain of g^{-1} is $[1/4, 1)$ and its codomain is $[0, \infty)$. Collating this information, the inverse is defined as

$$\mathrm{g}^{-1} : \begin{cases} [1/4, 1) \mapsto [0, \infty) \\ x \mapsto \sqrt{\dfrac{1-4x}{x-1}} \end{cases}$$

3658. In algebra, the symmetry statement is

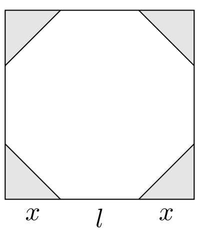
$$\mathrm{f}'(a+x) = \mathrm{f}'(a-x).$$

Using the reverse chain rule, we integrate to give

$$\begin{aligned} \mathrm{f}(a+x) &= -\,\mathrm{f}(a-x) + c \\ \implies \mathrm{f}(a+x) + \mathrm{f}(a-x) &= c. \end{aligned}$$

Hence, $\mathrm{f}(a+x) + \mathrm{f}(a-x)$ is constant, as required.

3659. We take the possibility space as the square below, which is repeated symmetrically.



By Pythagoras, the side length l of the octagon is $l = \sqrt{2}x$, so $x + x + \sqrt{2}x = 1$. Hence,

$$x = \frac{1}{2+\sqrt{2}}.$$

The area shaded above is therefore $2x^2 = 3 - 2\sqrt{2}$. Since the possibility space has area 1, this gives the probability directly: $3 - 2\sqrt{2} \approx 17\%$.

3660. Squaring the equations and adding them, the right-hand sides sum to 1 by the first Pythagorean trig identity. This gives

$$(ax+by)^2 + (bx-ay)^2 = 1.$$

Multiplying out, this is

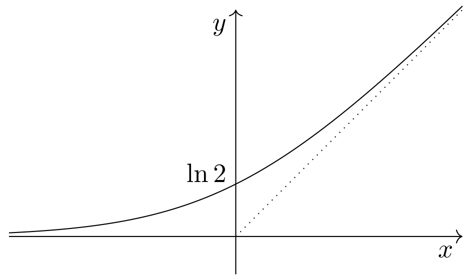
$$a^2x^2 + 2abxy + b^2y^2 + b^2x^2 - 2abxy + a^2y^2 = 1.$$

The cross terms cancel. Factorising the others,

$$\begin{aligned} (a^2+b^2)(x^2+y^2) &= 1 \\ \implies x^2 + y^2 &= \frac{1}{a^2+b^2}. \end{aligned}$$

The denominator on the RHS is non-zero, so the RHS is a well defined, positive constant r^2. Hence, the equations define a circle in the (x, y) plane, centred at the origin.

3661. As $x \to -\infty$, $y \to \ln 1 = 0$. This means the x axis is an asymptote. As $x \to \infty$, e^x becomes very large compared to 1. Hence, $y \to \ln(e^x) = x$. So, the line $y = x$ is an asymptote. The curve lies above both asymptotes, and crosses the y axis at $y = \ln 2$. So, the graph is



3662. Over a suitable domain/codomain, $y = \arctan x$ can be written as $x = \tan y$. Differentiating with respect to y,

$$\begin{aligned} x &= \tan y \\ \implies \frac{dx}{dy} &= \sec^2 y. \end{aligned}$$

Reciprocating, then using the second Pythagorean trig identity,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sec^2 y} \\ &\equiv \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2}. \end{aligned}$$

Hence, $\dfrac{d}{dx}(\arctan x) \equiv \dfrac{1}{1+x^2}$, as required.

3663. There is a common factor of $x^{0.1}$. Hence, $x = 0$ is a root. Dividing through by $x^{0.1}$ leaves a quadratic in $x^{0.3}$:

$$\begin{aligned} &x^{0.6} + x^{0.3} - 72 = 0 \\ \Longrightarrow\ &\left(x^{0.3} - 8\right)\left(x^{0.3} + 9\right) = 0 \\ \Longrightarrow\ &x^{0.3} = 8, -9. \end{aligned}$$

Since 0.3 is $\frac{3}{10}$, which has an even denominator, we reject the negative root. Solving $x^{0.3} = 8$,

$$x = 8^{\frac{10}{3}} = 1024.$$

So, the solution is $x = 0, 1024$.

3664. The second derivative h$''$ is quadratic. As seen in the diagram, it is positive everywhere.

So, h$'$ is increasing everywhere. Hence, there can be at most one value of x for which $\mathrm{h}'(x) = 0$.

The curve $y = \mathrm{h}(x)$, therefore, has a maximum of one turning point, so cannot have more than two real roots, as required.

——— Nota Bene ———

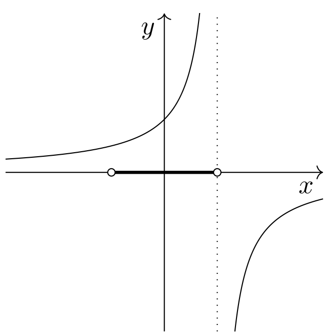
In fact, being a quartic, the curve $y = \mathrm{h}(x)$ must have *exactly* one turning point, but this fact isn't necessary for the proof in question.

3665. (a) The function is only well defined if the sum to infinity converges. For a GP, this occurs if the common ratio, in this case x, has $|x| < 1$. So, the largest real domain is $(-1, 1)$.

(b) If $x \in (-1, 1)$, then we can use the geometric sum formula, which gives

$$\mathrm{f}(x) = S_\infty = \frac{1}{1-x}.$$

The graph of $y = \dfrac{1}{1-x}$ is

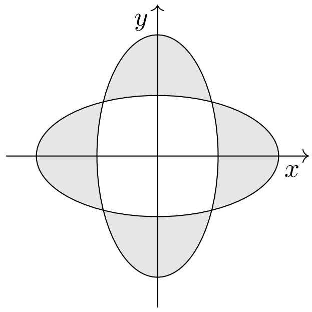


The domain is marked on the x axis. Over this domain, the range of f is $(1, \infty)$.

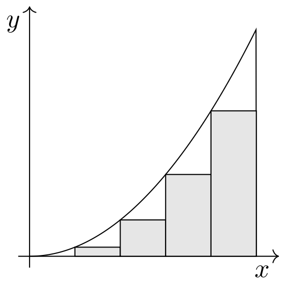
3666. The boundary equation is satisfied where either $x^2 + 4y^2 - 4 =$ or $4x^2 + y^2 - 4 = 0$. Rearranging these, we have

$$\begin{aligned} x^2 + (2y)^2 &= 4, \\ (2x)^2 + y^2 &= 4. \end{aligned}$$

These are two ellipses, which are reflections of each other in the line $y = x$. For the original product to be negative, we need the points inside one ellipse, but outside the other. This is



3667. (a) The scenario, with e.g. $n = 5$ and the top-left corners placed on the curve, is



The height of the rth rectangle (note that the first has zero height) is given by $(r-1)^2$. So, the estimate is

$$A_{\text{under}} = \sum_{r=1}^{n} \frac{(r-1)^2}{n^3} = \sum_{r=0}^{n-1} \frac{r^2}{n^3}.$$

(b) In each sum, we can take out a factor of n^3, which is constant. The estimate results give

$$\begin{aligned} &A_{\text{under}} < A < A_{\text{over}} \\ \therefore\quad &\frac{1}{n^3}\sum_{r=0}^{n-1} r^2 < A < \frac{1}{n^3}\sum_{r=1}^{n} r^2. \end{aligned}$$

On the LHS, the first term in the series is zero, so we can rewrite as

$$\frac{1}{n^3}\sum_{r=1}^{n-1} r^2 < A < \frac{1}{n^3}\sum_{r=1}^{n} r^2.$$

We can now use the given result regarding the sum of squared integers, the first $n-1$ on the

LHS and the first n on the RHS:

$$\begin{aligned}&\frac{1}{n^3}\cdot\tfrac{1}{6}(n-1)n\big(2(n-1)+1\big) < A\\ &\qquad < \frac{1}{n^3}\cdot\tfrac{1}{6}n(n+1)(2n+1)\\ \Longrightarrow\ &\frac{(n-1)\left(n-\frac{1}{2}\right)}{n^2} < 3A < \frac{(n+1)\left(n+\frac{1}{2}\right)}{n^2}.\end{aligned}$$

(c) As $n\to\infty$, both bounds tend to 1, as the leading n^2 terms dominate the others. Hence, by the squeeze theorem, $3A\to 1$. This gives $A = 1/3$, in agreement with integral calculus.

3668. The possibility space is 5^4 equally likely outcomes. Successful outcomes have four different colours. There are ${}^5\mathrm{C}_4$ ways of selecting four colours, and 4! orders of them. So,

$$p=\frac{{}^5\mathrm{C}_4\times 4!}{5^4}=\tfrac{24}{125}.$$

— Alternative Method —

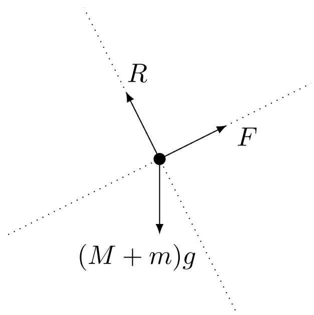
Colour one region wlog. For success, the second region must be a different colour, probability $4/5$. And the third must differ from the first two, with probability $3/5$. And the last must be different from all three, with probability $2/5$. So, the probability p that no two are the same is

$$p=\tfrac{4}{5}\cdot\tfrac{3}{5}\cdot\tfrac{2}{5}=\tfrac{24}{125}.$$

3669. Since the workman is moving at constant speed, the average frictional force on his feet will be the same as if he were standing still.

(a) On level ground, there are no horizontal forces, so $F=0$.

(b) The force diagram, considering workman and load as a single object of mass $M+m$, is



Resolving up the slope, $F=\frac{1}{2}(M+m)g$ N.

3670. Rewriting the integrand,

$$\begin{aligned}&\frac{x^2+x}{x-1}\\ \equiv\ &\frac{x(x-1)+2(x-1)+2}{x-1}\\ \equiv\ &x+2+\frac{2}{x-1}.\end{aligned}$$

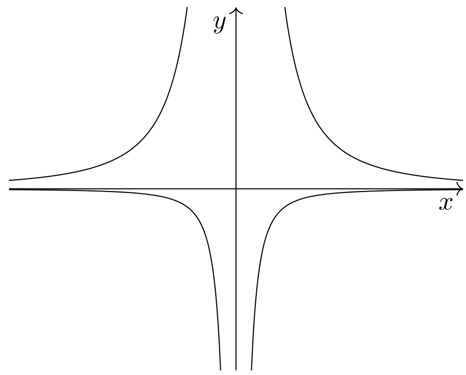
Carrying out the integral,

$$\begin{aligned}&\int_{-1}^{0} x+2+\frac{2}{x-1}\,dx\\ &=\left[\tfrac{1}{2}x^2+2x+2\ln|x-1|\right]_{-1}^{0}\\ &=2\ln|-1|-\left(\tfrac{1}{2}-2+2\ln|-2|\right)\\ &=\tfrac{3}{2}-2\ln 2\\ &=\tfrac{3}{2}-\ln 4,\ \text{as required.}\end{aligned}$$

3671. The relation is a quadratic in x^2y:

$$\begin{aligned}&x^4y^2-9x^2y-10=0\\ \Longrightarrow\ &(x^2y-10)(x^2y+1)=0\\ \Longrightarrow\ &x^2y=10,-1\\ \Longrightarrow\ &y=\frac{10}{x^2},-\frac{1}{x^2}.\end{aligned}$$

Each of these is a squared reciprocal graph. The former is above the y axis, the latter below. So, the curve is



3672. By the quotient rule,

$$\begin{aligned}&y=\frac{100}{x^2+9}\\ \Longrightarrow\ &\frac{dy}{dx}=-\frac{200x}{(x^2+9)^2}.\end{aligned}$$

So, at $\left(p,\frac{100}{p^2+9}\right)$, the equation of the tangent is

$$y-\frac{100}{p^2+9}=-\frac{200p}{(p^2+9)^2}(x-p).$$

Substituting in the point $\left(\frac{57}{8},0\right)$,

$$\begin{aligned}&-\frac{100}{p^2+9}=-\frac{200p}{(p^2+9)^2}\left(\tfrac{57}{8}-p\right)\\ \Longrightarrow\ &p^2+9=2p\left(\tfrac{57}{8}-p\right)\\ \Longrightarrow\ &p=\tfrac{3}{4},4.\end{aligned}$$

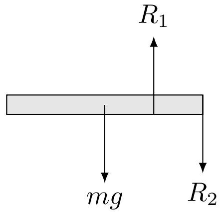
3673. (a) The function $x\mapsto\tan x$ has period π. Hence, $x\mapsto\tan\frac{1}{2}x$ has period 2π.

(b) The function $x \mapsto \operatorname{cosec} x$ has period 2π. So, $x \mapsto \operatorname{cosec} 2x$ has period π. The period of the sum is the lowest common multiple of the two periods, which is 2π.

3674. Writing $e^{x+y} \equiv e^x e^y$, we factorise:

$$\begin{aligned} & e^{x+y} + 1 = e^x + e^y \\ \Longrightarrow\ & e^x e^y - e^x - e^y + 1 = 0 \\ \Longrightarrow\ & (e^x - 1)(e^y - 1) = 0 \\ \Longrightarrow\ & e^x = 1 \text{ or } e^y = 1 \\ \Longrightarrow\ & x = 0 \text{ or } y = 0, \text{ as required.} \end{aligned}$$

3675. The force diagram is



Vertically, $R_1 - R_2 - mg = 0$. Taking moments around the right-hand end, $R_1 - \frac{1}{2}(1+k)mg = 0$. Subtracting the former from twice the latter,

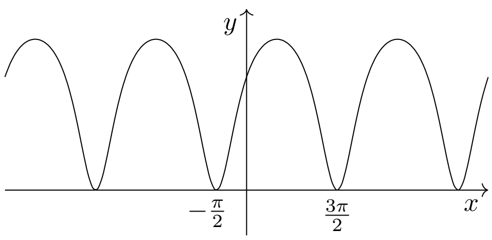
$$\begin{aligned} & 2R_1 - (1+k)mg - (R_1 - R_2 - mg) = 0 \\ \Longrightarrow\ & R_1 + R_2 = kmg, \text{ as required.} \end{aligned}$$

3676. (a) The denominator cannot be zero, as the range of the sine function is $[-1, 1]$.

(b) The range of the numerator is $[0, 2]$ and of the denominator is $[1, 3]$. These are maximised and minimised at the same x values, giving the range of the fraction as $[0/1, 2/3]$, which is $[0, 2/3]$.

(c) The first maximum is $x = \pi/2$, and the first minimum is $x = 3\pi/2$. At these, the second derivative is $-1/9$ and 1. Since $|-1/9| < 1$, the rate of downturn at the maxima is (much) less than the rate of upturn at the minima.

(d) Putting the above together, graph G is



3677. The first face can be chosen wlog. The probability that the next face shares a border with the first is $5/11$. Once two adjacent faces have been chosen, there are now two successful locations for the third face. So, the probability is

$$p = \tfrac{5}{11} \cdot \tfrac{2}{10} = \tfrac{1}{11}.$$

——— Alternative Method ———

For a combinatorial approach, there are ${}^{12}\mathrm{C}_3$ ways of selecting three faces. This is the possibility space. The number of successful outcomes is the number of vertices of a dodecahedron. This is 20. So, the probability is

$$p = \frac{20}{{}^{12}\mathrm{C}_3} = \tfrac{1}{11}.$$

3678. Consider separately limits $x \to 2^+$ and $x \to 2^-$, in which the notation means "x tends to 2 from above" and "x tends to 2 from below". For $x > 2$, the mod function is inactive, so

$$\lim_{x\to 2^+} \frac{x^2 - 2x}{|x-2|} = \lim_{x\to 2^+} \frac{x(x-2)}{x-2} = x.$$

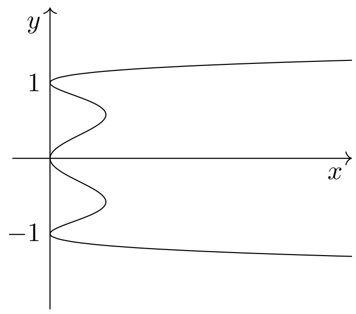
For $x < 2$, the mod function is active, so

$$\lim_{x\to 2^-} \frac{x^2 - 2x}{|x-2|} = \lim_{x\to 2^-} \frac{x(x-2)}{-(x-2)} = -x.$$

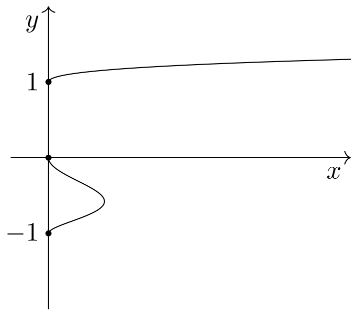
Since these are not identical, we cannot define the limit (as is done in in $x \to 2$) without reference to the direction in which the limit is being taken.

3679. (a) The boundary equation $y(y+1)(y-1) = 0$ has roots $y = 0, \pm 1$. The solution to the inequality is then $(-\infty, -1) \cup (0, 1)$.

(b) Having factorised, we square both sides, giving $x = y^2(y+1)^2(y-1)^2$. This is a positive sextic graph, with double roots (points of tangency with the y axis) at $y = 0, \pm 1$. This gives



However, by squaring we have introduced new solution points. Where $y^3 - y < 0$, there are none, as $\sqrt{x}$ must be non-negative. Hence, the graph $\sqrt{x} = y^3 - y$ is



3680. The implication is

$$\mathrm{f}''(x) \equiv \mathrm{g}''(x) \impliedby \mathrm{f}'(x) \equiv \mathrm{g}'(x).$$

① follows from ②, by differentiation. However, ② does not follow from ①, since, in integrating, a constant is introduced. A counterexample is

$$\begin{aligned} \mathrm{f}(x) &= x^2 + x, \\ \mathrm{g}(x) &= x^2 + 3x. \end{aligned}$$

The second derivatives are both 2, but the first derivatives differ.

3681. For small x, $\sin x \approx x$ (assuming radian measure). We expand binomially, ignoring terms in x^3 or higher. The numerator is

$$\begin{aligned} &(1+2x)^7 \\ &\equiv 1 + {}^7\mathrm{C}_1(2x) + {}^7\mathrm{C}_2(2x)^2 + ... \\ &\equiv 1 + 14x + 84x^2 + ... \end{aligned}$$

The denominator (seen multiplicatively) is

$$\begin{aligned} &(1+3x)^{-4} \\ &= 1 + (-4)(3x) + \tfrac{(-4)(-5)}{2!}(3x)^2 + ... \\ &\equiv 1 - 12x + 90x^2 + ... \end{aligned}$$

So, the full expansion is

$$\begin{aligned} &(1 + 14x + 84x^2 + ...)(1 - 12x + 90x^2 + ...) \\ &= 1 + 2x + 6x^2 + ..., \text{ as required.} \end{aligned}$$

3682. (a) This is false. A counterexample is $\mathrm{f}(x) = \sin x$ and $\mathrm{g}(x) = \cos x$. These attain their maxima at different x values, reducing the range. For them, the range of the sum is $\left[-\sqrt{2}, \sqrt{2}\right]$.

(b) This is true. Since the individual functions produce outputs in $[-1, 1]$, it is impossible for their sum to produce an output which is not in $[-2, 2]$.

3683. Solving for intersections,

$$\begin{aligned} px^2 + q\big(x^2\big)^2 &= 1 \\ \implies qx^4 + px^2 - 1 &= 0. \end{aligned}$$

This is a quadratic in x^2 with $\Delta = p^2 + 4q$. So, if we choose p and q such that $p^2 + 4q < 0$, there will be no intersections. For example, $p = 1, q = -1$.

3684. The velocities are

$$\begin{aligned} \dot{x} &= 3\cos 3t, \\ \dot{y} &= -4\sin 4t. \end{aligned}$$

Setting both of these to zero (for $t \geq 0$),

$$\begin{aligned} 3\cos 3t = 0 &\implies t \in \{\tfrac{\pi}{6}, \tfrac{3\pi}{6}, \tfrac{5\pi}{6}, ...\}, \\ -4\sin 4t = 0 &\implies t \in \{\tfrac{\pi}{4}, \tfrac{2\pi}{4}, \tfrac{3\pi}{4}, ...\}. \end{aligned}$$

At e.g. $t = \frac{\pi}{2}$, both components of the velocity are zero, hence the particle is momentarily at rest.

3685. (a) The total frequency must be 1 million. So,

$$\begin{aligned} k\int_0^1 x^3 - x^7\,dx &= 10^6 \\ \implies k \times \tfrac{1}{8} &= 10^6 \\ \implies k &= 8 \times 10^6. \end{aligned}$$

(b) The modal value has the maximum frequency density. So, we set

$$\begin{aligned} \mathrm{h}'(x) = k\big(3x^2 - 7x^6\big) &= 0 \\ \implies x &= 0, \pm\sqrt[4]{3/7}. \end{aligned}$$

The negative root is outside the domain, and $x = 0$ is a minimum. So, the mode is 0.809 (3sf).

(c) For the median m, we require

$$\begin{aligned} 8 \times 10^6 \int_0^m x^3 - x^7\,dx &= \tfrac{1}{2} \times 10^6 \\ \implies \Big[\tfrac{1}{4}x^4 - \tfrac{1}{8}x^8\Big]_0^m &= \tfrac{1}{16} \\ \implies 2m^8 - 4m^4 + 1 &= 0. \end{aligned}$$

This is a quadratic in m^4. The formula gives

$$m^4 = \tfrac{4 \pm \sqrt{16-8}}{4} = 1 \pm \tfrac{\sqrt{2}}{2}.$$

For the larger of these, $m > 1$. So, we take the smaller value. According to the model,

$$m = \sqrt[4]{1 - \tfrac{\sqrt{2}}{2}}.$$

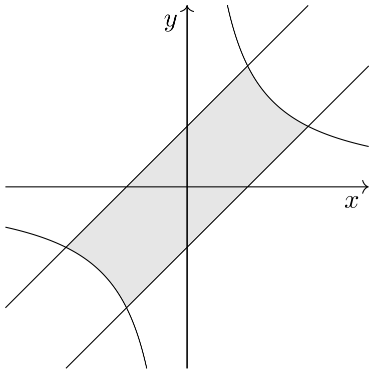
3686. Call the lengths $(a, a, 2b)$. Then we can split the triangle into two symmetrical $\big(a, b, \sqrt{a^2 - b^2}\big)$ right-angled triangles. This gives $b\sqrt{a^2 - b^2} = 360$ and $a + b = 50$. Substituting for a,

$$\begin{aligned} b\sqrt{(50-b)^2 - b^2} &= 360 \\ \implies b^2(2500 - 100b) &= 129600 \\ \implies b^3 - 25b^2 + 1296 &= 0 \\ \implies b &= 9, 8 \pm 4\sqrt{13}. \end{aligned}$$

Taking the integer root $b = 9$, the triangle has side lengths $(18, 41, 41)$.

3687. (a) The mod graph is $x - y = \pm 1$, which is a pair of straight lines $x = y \pm 1$. Subbing into $xy = 2$ gives $(y \pm 1)y = 2$, so $y = -2, -1, 1, 2$. The points of intersection are $(\pm 2, \pm 1)$ and $(\pm 1, \pm 2)$.

(b) The boundary equations are $y = {}^2/_x$, which is a standard reciprocal graph (hyperbola), and $x - y = \pm 1$, which is a pair of straight lines. The region R contains the origin:



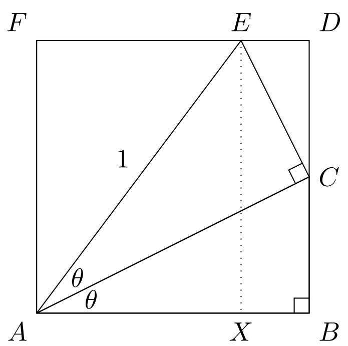
(c) To find the area, we integrate ${}^2/_x$ between 1 and 2. This gives $\ln 4$. In the first quadrant, we then add and subtract triangles of area ${}^1/_2$, and add a square of area 1, giving $\ln 4 + 1$. The second quadrant then contributes an area ${}^1/_2$. Doubling via rotational symmetry, the area of region R is $2(\ln 4 + {}^3/_2)$, which is $3 + \ln 16$.

3688. The derivative is

$$\begin{aligned} \frac{dy}{dx} &= \tfrac{1}{2}(1 - x^n)^{-1/2} \cdot nx^{n-1} \\ &\equiv \frac{nx^{n-1}}{2\sqrt{1 - x^n}}. \end{aligned}$$

At $x = 1$, the denominator is zero, so the gradient is undefined: the tangent is parallel to the y axis. At $x = 0$, the denominator is non-zero, but, since $n \in (0, 1)$, the index $n - 1 < 0$, so the numerator is undefined: the tangent is parallel to the y axis.

3689. Drop a perpendicular from E to AB, to produce AX, whose length is $\sin 2\theta$:



With hypotenuse 1, $|AC| = \cos\theta$ and $|EC| = \sin\theta$. This gives $|BC| = \sin\theta\cos\theta$. Also, $\angle DCE = \theta$, so $|CD| = \sin\theta\cos\theta$. Hence, $|BD| = 2\sin\theta\cos\theta$. Since $|EX| = |BD|$, $\sin 2\theta \equiv 2\sin\theta\cos\theta$. □

3690. (a) The natural logarithm has domain $(0, \infty)$ and is not defined at $x = 0$. So, any definition of $x \ln x$ at $x = 0$ requires consideration of limits.

(b) By the product rule, $f'(x) = \ln x + 1$. And log functions are increasing everywhere, so, on the domain $(0, {}^1/_e)$,

$$\begin{aligned} &\ln x < \ln \tfrac{1}{e} \\ \implies &\ln x < -1. \end{aligned}$$

Hence, on $(0, {}^1/_e)$, $f'(x) = \ln x + 1 < 0$.

(c) f is decreasing (as x increases) on $(0, {}^1/_e)$. So, as we let x tend to 0 from above (that is, as x decreases), the value of f must increase. It is increasing from a negative value, so cannot diverge to $-\infty$. We also know that, on $(0, {}^1/_e)$, the value of $x \ln x$ is negative. So, the limit cannot diverge to $+\infty$. This proves the result.

— Nota Bene —

In fact, $\lim_{x \to 0^+} x \ln x = 0$.

3691. Differentiating with respect to time,

$$\begin{aligned} &x = c\sin\omega t + d\cos\omega t \\ \implies &\dot{x} = c\omega\cos\omega t - d\omega\sin\omega t \\ \implies &\ddot{x} = -c\omega^2\sin\omega t - d\omega^2\cos\omega t. \end{aligned}$$

Substituting into $6x + a$,

$$\begin{aligned} &6c\sin\omega t + 6d\cos\omega t - c\omega^2\sin\omega t - d\omega^2\cos\omega t \\ \equiv\ &\big(6 - \omega^2\big)\big(c\sin\omega t + d\cos t\big). \end{aligned}$$

If $\omega = \pm\sqrt{6}$, then this will equal zero, satisfying $6x + a = 0$, irrespective of the values of c and d. The positive value of ω is $\sqrt{6}$.

3692. We need to restrict the domain and codomain of f such that f is one-to-one. Looking for SPs,

$$\begin{aligned} &f'(x) = \tfrac{1}{x-1} - \tfrac{1}{x} = 0 \\ \implies &\frac{1}{x(x-1)} = 0. \end{aligned}$$

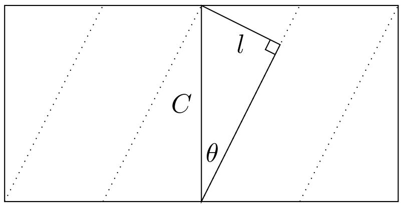
This has no roots, so the curve has no SPs. Its domain is $(1, \infty)$. To establish the range, we note that, as $x \to 1^+$, $\ln(x - 1) \to -\infty$ and therefore $f(x) \to -\infty$. Rearranging to

$$f(x) = \ln\left(\frac{x-1}{x}\right) \equiv \ln\left(1 - \frac{1}{x}\right),$$

we see that, as $x \to \infty$, $1 - \frac{1}{x} \to 1^-$, so $f(x) \to 0^-$. Hence, the range of f is $(-\infty, 0)$.

To summarise, for f invertible, the largest possible real domain is $(1, \infty)$ and the largest possible real codomain is $(-\infty, 0)$.

3693. Unwrap the cylinder, cutting it lengthwise and flattening it to a rectangle. The dotted lines show the edges of the strip of paper.



Using the right-angled triangle shown, elementary trigonometry gives $\theta = \arcsin \frac{l}{C}$.

3694. (a) At $x = 0$, $y = -1$. Setting $y = 0$ gives

$$\begin{aligned} x^2 - x - 1 &= 0 \\ \implies x &= \tfrac{1}{2}\big(1 \pm \sqrt{5}\big). \end{aligned}$$

(b) Using the product rule,

$$\begin{aligned} \frac{dy}{dx} &= (2x-1)e^x + (x^2 - x - 1)e^x \\ &\equiv (x^2 + x - 2)e^x. \end{aligned}$$

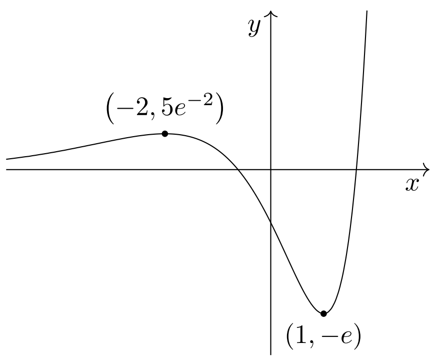
The exponential factor is always positive, so, for SPs, $x^2 + x - 2 = 0 \implies x = -2, 1$. The second derivative is

$$\begin{aligned} \frac{d^2y}{dx^2} &= (2x+1)e^x + (x^2 + x - 2)e^x \\ &\equiv (x^2 + 3x - 1)e^x. \end{aligned}$$

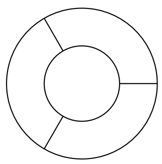
At $x = -2$, the second derivative is $-3e^{-2} < 0$, so $\big(-2, 5e^{-2}\big)$ is a local maximum. At $x = 1$, the second derivative is $3e > 0$, so $(1, -e)$ is a local minimum.

(c) As $x \to \infty$, $y \to \infty$.
As $x \to -\infty$, the exponential factor tends to zero and the polynomial factor tends to +ve infinity. Exponentials dominate, so $y \to 0^+$.

(d) Putting the above together, the graph is



3695. The classic counterexample is as follows:



This clearly requires four colours to shade it, since every region borders every other.

3696. A curve has a horizontal asymptote if, when x tends to either plus or minus infinity, y tends to a constant.

(a) True. Since the degree of the denominator is larger than the degree of the numerator, the x axis is an asymptote.

(b) True. Degree of numerator and denominator are the same: when $x \to \pm\infty$, the $+1$ becomes negligible and $y \to 1$.

(c) False. The degree of the numerator is larger, so, as $x \to \pm\infty$, $y \to \pm\infty$.

3697. The double-angle formula $\cos 2x \equiv 1 - 2\sin^2 x$ is $\sin^2 x \equiv \frac{1}{2}(1 - \cos 2x)$. Replacing x by nx,

$$\begin{aligned} &\int_0^\pi \sin^2(nx)\,dx \\ &= \tfrac{1}{2}\int_0^\pi 1 - \cos(2nx)\,dx \\ &= \tfrac{1}{2}\Big[x - \tfrac{1}{2n}\sin(2nx)\Big]_0^\pi \\ &= \tfrac{1}{2}(\pi) - \tfrac{1}{2}(0) \\ &= \tfrac{\pi}{2}, \text{ as required.} \end{aligned}$$

3698. 4 can be obtained by rolling

- four, then four sixes from four rolls,
- five, then four sixes from five rolls,
- six, then three sixes from six rolls.

Each initial outcome has probability $1/6$. Then, we use $B(n, 1/6)$, with $n = 4, 5, 6$. This gives

$$\begin{aligned} \mathbb{P}(4) &= \tfrac{1}{6}\Big(\tfrac{1}{6}^4 + {}^5\mathrm{C}_4 \cdot \tfrac{1}{6}^4 \cdot \tfrac{5}{6} + {}^6\mathrm{C}_3 \cdot \tfrac{1}{6}^3 \cdot \tfrac{5}{6}^3\Big) \\ &= 0.009595 \\ &\approx 1\%, \text{ as required.} \end{aligned}$$

3699. (a) Substituting the proposed equations into the LHS of the Cartesian equation,

$$\begin{aligned} &\sqrt{\cos^4 t} + \sqrt{\sin^4 t} \\ &\equiv \cos^2 t + \sin^2 t. \end{aligned}$$

By the first Pythagorean trig identity, this is equal to 1, as required.

(b) Using the parametric differentiation formula,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{4\sin^3 t \cos t}{-4\cos^3 t \sin t} \\ &\equiv -\frac{\sin^2 t}{\cos^2 t} \\ &\equiv -\tan^2 t, \text{ as required.} \end{aligned}$$

(c) At $x = 1/16$, $\cos t = 1/2$, so $t = \pi/3$. Hence, the gradient of the tangent is $-\tan^2 \pi/3 = -3$. So, the equation of the tangent is

$$\begin{aligned} & y - \tfrac{9}{16} = -3\big(x - \tfrac{1}{16}\big) \\ \Longrightarrow\ & 12x + 4y = 3. \end{aligned}$$

3700. Given equilibrium, $a+3+1 = 0$ and $-5+b-4 = 0$, so $a = -4$ and $b = 9$. So, the lines of action of the first two forces are $y = \frac{5}{4}x - 3$ and $y = 3x - 3$. These two meet at $(0, -3)$. The line of action of the third force must be concurrent with the first two. It has gradient -4, so its equation is $y = -4x - 3$. Setting $x = -1$ gives $c = 1$.

——— End of 37th Hundred ———